FOUR-DIMENSIONAL RIEMANNIAN STANILOV-VIDEV MANIFOLDS

Veselin Totev Videv
Dept.”Informatics and Mathematics”
Trakia University, Stara Zagora, Bulgaria, EU

ABSTRACT

In the present note we characterize the four-dimensional Riemannian manifolds for which any skew-symmetric Stanilov operator with respect to any plane in the tangent space to the manifold commute with the corresponding generalized Jacobi operator, at any point of the manifold. These class of manifolds Gilkey called Stanilov-Videv manifolds.

MSC(2000). 53C20

Let (M,g) be an n-dimensional Riemannian manifold with a metric tensor g and curvature tensor R, let p be a point of M, and let M_p be the tangent space to the manifold at this point. Then we can consider the following curvature operators:

Jacobi operator

$$R_X(u)=R(u,X,X),$$

defined for any tangent vector $X\in M_p$, at any point $p\in M$[1],

skew-symmetric Stanilov operator

$$S_\alpha(u)=S_{X,Y}(u)=R(X,Y,u),$$

defined for an arbitrary orthonormal basis X,Y in the plane $\alpha\in M_p$, at a point $p\in M$[4],

generalized Jacobi operator

$$R_\pi(u)=\Sigma_{i=1,m}R_{Xi}(u),$$

defined for an arbitrary m-dimensional tangent subspace $\pi\subset M_p$, at a point $p\in M$, where $\{X_i\}_{i=1,m}$ $(m<n)$ is an orthonormal basis in π[2].

Having in mind our preliminary results[7], Peter Gilkey introduces the following definitions[3]**:**

Definition 1. *An n-dimesional Riemannian manifold* (M,g) *is Stanilov-Videv manifold if for any two-dimensional plane* $\alpha\in M_p$, *at any point* $p\in M$, *the following equality holds*:

(1) $$S_\alpha\circ R_\alpha= R_\alpha\circ S_\alpha .$$

Further in the present paper we’ll characterize the four-dimensional Stanilov-Videv manifolds. First we replace the commutative condition (1) with the equivalent conditions $S_\alpha\circ R_\alpha$ to be a skew-symmetric curvature operator and we will show that if (1) holds, then (M,g) is an *Einstein Riemannian manifold*, which means that[6]:

(2) $$\rho=cg ,$$

where c is a constant and ρ is the Ricci tensor on the manifold (M,g).

Let $\{e_1,e_2,\ldots,e_n\}$ be an orthonormal basis in the tangent space M_p, at a point $p\in M$. If α is a two-dimensional plane, spanned from the tangent vectors e_1 and e_2, then:

$$(S_\alpha)=\begin{pmatrix} 0 & -K_{12} & -R_{2113} & -R_{2114} \\ K_{12} & 0 & R_{1223} & R_{1224} \\ R_{2113} & -R_{1223} & 0 & R_{1234} \\ R_{2114} & -R_{1224} & -R_{1234} & 0 \end{pmatrix},$$

$$(R_\alpha)=\begin{pmatrix} K_{12} & 0 & R_{1223} & R_{1224} \\ 0 & K_{12} & R_{2113} & R_{2114} \\ R_{1223} & R_{2113} & K_{13}+K_{23} & R_{3114}+R_{3224} \\ R_{1224} & R_{2114} & R_{3114}+R_{3224} & K_{14}+K_{24} \end{pmatrix}.$$

From here, multiplying, for the entries of the operator matrix $A:=S_\alpha\circ R_\alpha$, with respect to the orthonormal basis $\{e_1,e_2,\dots,e_n\}$, we obtain:

(3) $$a_{11}=-R_{2113}R_{1223}-R_{2114}R_{1224}$$
$$a_{22}=R_{1223}R_{2113}+R_{1224}R_{2114}$$
$$a_{33}=R_{1234}(R_{3114}+R_{3224})$$
$$a_{44}=-R_{1234}(R_{3114}+R_{3224})$$
$$a_{12}=-K^2_{12}-R^2_{2113}-R^2_{2114}$$
$$a_{21}=K^2_{12}+R^2_{1223}+R^2_{1224}$$
$$a_{13}=-K_{12}R_{2113}-(K_{13}+K_{23})R_{2113}-R_{2114}(R_{3114}+R_{2114})$$
$$a_{31}=K_{12}R_{2113}+R_{1234}R_{1224}$$
$$a_{14}=-K_{12}R_{2114}-R_{2113}(R_{3114}+R_{3224})-R_{2114}(K_{14}+K_{24})$$
$$a_{41}=K_{12}R_{2114}-R_{1234}R_{1223}$$
$$a_{23}=(K_{12}+K_{13}+K_{23})R_{1223}+R_{1224}(R_{3114}+R_{3224})$$
$$a_{32}=-K_{12}R_{1223}+R_{2114}.R_{1234}$$
$$a_{24}=K_{12}R_{1224}+R_{1223}(R_{3114}+R_{3224})+(K_{14}+K_{24})R_{1224}$$
$$a_{42}=-K_{12}.R_{1224}-R_{1234}R_{2113}$$
$$a_{34}=R_{2113}R_{1224}-R_{1223}R_{2114}+(K_{14}+K_{24})R_{1234}$$
$$a_{43}=R_{2114}R_{1223}-R_{1224}R_{2113}-R_{1234}(K_{13}+K_{23}).$$

Since A is a skew-symmetric matrix, then we have the equalities:

(4) $$a_{ii}=0\ ,\ a_{ij}=-a_{ji}\,,$$

for any indices $i,j=1,2,3,4$, and using more exactly that $a_{12}=-a_{21}$, according to the respective expression in (3), we obtain the equality:

(5) $$R^2_{2113}+R^2_{2114}=R^2_{1223}+R^2_{1224}\ .$$

Let we choose $\{e_1, e_2, e_3, e_4\}$ to be an orthonormal eigenvector basis of the Jacobi operator R_{e1}. Then from this condition we obtain that:

(6) $$R_{2113}=R_{2114}=R_{3114}=0\ .$$

Now from (5) and (6) we get:

(7) $$R_{2113}=R_{2114}=R_{1223}=R_{1224}=0\ .$$

Since the hypothesis (1) is true for any plane $\alpha=e_i\wedge e_j$, similarly to the way of obtaining (7), we get the following equalities:

$$R_{2113}=R_{2114}=R_{1223}=R_{1224}=R_{3112}=R_{3114}=R_{1332}$$
$$=R_{1334}=R_{4112}=R_{4113}=R_{1442}=R_{1443}=0.$$

From this we conclude that for the entries of the Ricci tensor is true

$$\rho_{12}=\rho_{13}=\rho_{14}\,,$$

which in turn gives us

$$\rho_{1x}=\rho(e_1,x)=0\ ,$$

for any unit tangent vector $x\in M_p$, such that $x\perp e_1$, at any point $p\in M$. This result according to the Herglotz theorem means that (M,g) is a four-dimensional Einstein Riemannian manifold[6], that was our result in [7], and in the sequel we'll continue these investigations to prove main result in the present paper.

Suppose again that $\{e_1,e_2,\dots,e_n\}$ is an arbitrary orthonormal basis in the tangent space M_p, at a point $p\in M$. Then from the hypothesis (1) we have the equality $a_{13}=-a_{31}$ and from the respective expressions in (3) we get the equation:

$$(K_{13}+K_{23})R_{2113}+R_{2114}(R_{3114}+R_{2114})=R_{1234}R_{1224}.$$

Since (M,g) is an Einstein Riemannian manifold, then $K_{13}=K_{23}$, and now the last equality gives us:

(8) $$(K_{13}+K_{14})R_{2113}+R_{2114}R_{3114}+R^2_{2114}=R_{1234}R_{1224}.$$

Since here $\{e_1,e_2,\dots,e_n\}$ is an orthonormal basis in the tangent space M_p, at an arbitrary point $p\in M$, then we can substitute e_2 with $-e_2$. In this way we get the following equality:

(9) $$-(K_{13}+K_{14})R_{2113}-R_{2114}R_{3114}+R^2_{2114}=-R_{1234}R_{1224}.$$

Summing (8) and (9), we obtain the equation:

$$R^2_{2114}=0.$$

From the last equality it follows that

$$R(x,y,y,z)=0,$$

for an arbitrary orthonormal triple of tangent vectors $x,y,z\in M_p$, at any point $p\in M$, because of $\{e_1,e_2,\dots,e_l\}$ is an orthonormal basis in the tangent space M_p. That means that we can apply the last equality for the orthonormal triple of tangent vectors $\{\frac{x+z}{\sqrt{2}},\ y,\ y,\ \frac{x-z}{\sqrt{2}}\}$ and so we get the equation:

$$R(\frac{x+z}{\sqrt{2}},\ y,\ y,\ \frac{x-z}{\sqrt{2}})=0,$$

from which it follows that

$$R(x+z,y,y,x-z)=0.$$

This equality is equivalent to the equation:

(10) $$K(x,y)=K(y,z),$$

for any orthonormal triple of tangent vectors $x,y,z\in M_p$, at any point $p\in M$, where K is a function of the sectional curvature of the manifold (M,g).

Since we proved above that (M,g) *is an Einstein four-dimensional manifold*, then at any point $p\in M$ there exists an orthonormal tangent Singer-Thorpe basis $\{e_1,e_2,e_3,e_4\}$ in the tangent space M_p, with respect to which the curvature tensor entries satisfy the following equalities[5]:

$$K_{12}=K_{34}=\lambda_1,\ K_{13}=K_{24}=\lambda_2,\ K_{14}=K_{23}=\lambda_3,$$

$$R_{1234}=\mu_1,\ R_{1342}=\mu_2,\ R_{1423}=\mu_3,$$

$$R_{1223}=R_{1443}=R_{1332}=R_{1442}=R_{1224}=R_{1334}=0,$$

where

$$\mu_1+\mu_2+\mu_3=0,$$

$$\lambda_1+\lambda_2+\lambda_3=\frac{\tau}{4},$$

τ is a scalar curvature of (M,g) and $\lambda_1=\max K$, $\lambda_3=\min K$. From here and (10) we get that $\max K=\min K$, which means that K is a constant at any point $p\in M$, which according to the Shour's theorem means that K is a global constant of the manifold (M,g)[6]. Then according to definition (M,g) is a four-dimensional Riemannian manifold of a constant sectional curvature and curvature tensor of the form

(11) $$R(x,y,z)=K.\big(g(y,z)x-g(x,z)y\big),\qquad K=const.;$$

for any tangent vectors $x,y,z\in M_p$, at any point $p\in M$[6].

Conversely, if (M,g) is a four-dimensional Riemannian manifold of a constant sectional curvature and curvature tensor of the form (11), then for any two-dimensional tangent plane $\alpha\in M_p$, at any pont $p\in M$, and for any orthonormal basis $\{e_1,e_2,e_3,e_4\}\in M_p$, we have the following curvature matrix

$$S_{\alpha} = \begin{pmatrix} 0 & -c^2 & 0 & 0 \\ c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_{\alpha} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 2c & 0 \\ 0 & 0 & 0 & 2c \end{pmatrix}.$$

From here we conclude that $S_\alpha \circ R_\alpha = R_\alpha \circ S_\alpha$, since these operators have the same matrixs:

$$\begin{pmatrix} 0 & -c^3 & 0 & 0 \\ c^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have proved our main result:

Theorem 1. *A four-dimensional Riemannian manifold* (M,g) *is Stanilov-Videv manifold if and only if* (M,g) *is a four-dimensional Riemannian space of constant sectional curvature.*

In the pseudo-Euclidean geometry the things are different because of Gilkey proved that *there exist pseudo-Riemannian Stanilov-Videv manifolds which are not pseudo-Euclidean manifolds of constant sectional curvature* [3].

REFERENCES

1. Gilkey P., A. Swann A, L. Vanheke. *Isoparametric geodesic spheres and a conjecture of Osserman cocerning the Jacobi operator.* Quart. J. Math. Oxford (1995), 299-320

2. Gilkey P., G.Stanilov, V.Videv. *Pseudo-Riemannian manifolds whose generalized Jacobi operator has a constant characteristic polynomial.* Journal of Geometry(Basel), 62(1998), 144-153

3. Gilkey. *The Geometry of Curvature Homogeneous Pseudo-Riemannian manifolds*". Word Scientific Publishing Co(2010).

1. 4. Ivanova R., G. Stanilov. *A skew-symmetric curvature operator in Riemannian geometry.* Symposia Gaussiana, Conf. A, Eds.: Behara / Fritsch / Lintz, Berlin, New York (1995), 391-395

4. Singer I. , J. Thorpe. *The curvature of* 4-*dimensional Einstein spaces.* Global Analisys. Papers in Honour of K. Kodaira. University of Tokio press, Princeton University Press (1969)

5. Stanilov G. *Differential geometry.* Sofia, Nauka i Izkustvo, 1998.

6. G.Stanilov, V.Videv. *On the commuting of curvature operators.* Mathematics and Education in Mathematics(2004), 176-180