ON THE POLAR SINGULARITIES OF LAGUERRE SERIES

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ABSTRACT

We consider a series in Lagurre polynomials with poles on the boundaries of their regions of convergence.

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Let $\alpha > -1$. The Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{+\infty}$ with parameter α can be defined by the equalities [1, p. 33, (5.12)]

$$L_n^{(\alpha)}(z) = \binom{n+\alpha}{n} \Phi(-n, \alpha+1; z), n = 0, 1, 2, \dots; z \in \mathbf{C},$$

where **C** is the complex plane and $\Phi(a, b; \zeta)$ is a confluent hypergeometric function.

For Laguerre polynomials a following representation holds [2, Th. 8.22.3] $(n \geq 1)$

(1) $$L_n^{(\alpha)}(z) = l^{(\alpha)}(z)\varphi_n^{(\alpha)}(z)\{1+\lambda_n^{(\alpha)}(z)\},$$

where $l^{(\alpha)}(z) = (2\sqrt{\pi})^{-1}\exp(z/2)(-z)^{-\alpha/2-1/4}$, $\varphi_n^{(\alpha)}(z) = n^{\alpha/2-1/4}\exp\{2n^{1/2}\sqrt{-z}\}$ and $\{\lambda_n^{(\alpha)}(z)\}_{n=1}^{\infty}$ are a holomorhpic functions in a region $G = \mathbf{C} \setminus [0, +\infty)$. Here

(2) $$\lambda_n^{(\alpha)}(z) = O(n^{-1/2}) \ (n \to \infty)$$

uniformly on every compact subset of *G*.

A series of kind

(3) $$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$$

we shall call Laguerre series.

Let $0 < \lambda < +\infty$. We define by $p(\lambda)$ the parabola with equation $y^2 = 4\lambda^2(x+\lambda^2)$. This parabola have a vertex at the point $(-\lambda^2, 0)$ and a focus at the origin. For every point $z \in p(\lambda)$ we have that $\mathrm{Re}(-z)^{1/2} = \lambda$. We denote by $\Delta(\lambda)$ the interior of the parabola $p(\lambda)$ and by $\Delta^*(\lambda)$ its exterior. Let $\Delta(0) = \varnothing$, $\Delta(\infty) = £$, $\Delta^*(0) = G$ and $\Delta^*(\infty) = \varnothing$.

Using the asymptotic formulas (1) and (2) can be established following proposition

Theorem 1. Let $\alpha > -1$ and $z_0 \in G$. If p is a real number and $a_n L_n^{(\alpha)}(z_0) = O(n^p) \ (n \to \infty)$, then the series (3) is absolutely uniformly convergent on every compact subset of the region $\Delta(\lambda_0)$, where $\lambda_0 = \mathrm{Re}\sqrt{-z_0}$.

From this statement it follows

Theorem 2. Let $\alpha > -1$ and $z_0 \in G$. If the series (3) converges for $z = z_0$, then the series (3) is absolutely and uniformly convergent on every compact subset of the region $\Delta(\lambda_0)$, where $\lambda_0 = \mathrm{Re}\sqrt{-z_0}$. The sum of (3) is a complex function holomorphic in $\Delta(\lambda_0)$.

Theorem 3. [1, (IV.2.1), (b)] Let $\alpha > -1$ and

(4) $$\lambda_0 = \max\{0, -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n|\}.$$

Then the series (3) is absolutely uniformly convergent on every compact subset of the region $\Delta(\lambda_0)$ and diverges in $\Delta^*(\lambda_0)$.

The equality (4) can be regarded as a *formula Cauchy-Hadamard type* for the Laguerre series. The main result in this paper is the following

Theorem 4. Let $\alpha > -1$, $z_0 \in G$ and $a_n L_n^{(\alpha)}(z_0) = o(n^p)$ $(n \to \infty)$, where p is a nonnegative integer. If $f(z)$ is the sum of the series (3) in the region $\Delta(\lambda_0)$, where $\lambda_0 = \mathrm{Re}(-z_0)$, and $f(z)$ has a pole of order *m* on the parabola $p(\lambda_0)$, then $m \leq 2p+1$.

Proof. Suppose that there is a point $z_1 \in p(\lambda_0)$ such that the holomorphic function $f(z)$ has a pole at z_1 of order *m* and $m > 2p+1$. Obviously

(5) $\qquad m \geq 2p+2$.

Since the Laguerre polynomials have no zeros outside real line, we can write that

$$h(z) = (z - z_1)^m f(z) = (z - z_1)^m \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z_0) \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)},$$

where $z \in \Delta_1(\lambda_0) = \Delta(\lambda_0) \setminus [0, +\infty)$. Let *s* be a nonnegative integer. Then

$$|h(z)| = |(z - z_1)^m f(z)| \leq |(z - z_1)^m \sum_{n=0}^{s} a_n L_n^{(\alpha)}(z)| + |(z - z_1)^m \sum_{n=s+1}^{\infty} a_n L_n^{(\alpha)}(z_0) \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)}|$$

$$= h_{s,1}(z) + h_{s,2}(z).$$

Let $\varepsilon > 0$. Then it follows that there exists a positive s_0 such that

$$|a_n L_n^{(\alpha)}(z_0)| < \varepsilon n^p$$

for $n > s_0$. For $n > s_0$ we have that

(6) $$h_{s,2}(z) \leq \varepsilon |z - z_1|^m \sum_{n=s+1}^{\infty} n^p \left| \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} \right|.$$

Let $z_1 = x_1 + iy_1$ and $y_1 > 0$. We define $\delta \in (0, \pi/2)$ by equality $\tan\delta = 2\lambda_0 y_1^{-1}$, where obviously $\lambda_0 = \mathrm{Re}(-z_1)$. If $0 \leq \varphi < \pi/2$ and $0 < r < dist(z_1, [0, +\infty))$, then denote by $D(z_1, r, \varphi)$ the set defined by the inequalities $0 < |z - z_1| \leq r$ and $|\arg(z_1 - z) - \pi/2 - \delta| \leq \varphi$.

If $y_1 < 0$, then denote by $D(z_1, r, \varphi)$ the image of the set $D(\overline{z_1}, r, \varphi)$ by the map $z \to \overline{z}$. If $z = -\lambda_0^2$, then we define $D(-\lambda_0^2, r, \varphi)$ by means of the inegualities $0 < |z + \lambda_0^2| \leq r < \lambda_0$ and $|\arg(z + \lambda_0^2)| \leq \varphi$.

It is easy to see that $D(z_1, r, \varphi)$ is a subset of the region $\Delta_1(\lambda_0)$ and $\overline{D}(z_1, r, \varphi) \cap [0, +\infty) = \varnothing$.

Suppose that φ and *r* are a fixed. Then there exists a constant *K* such that

(7) $\qquad |z - z_1| \leq K\{\lambda_0 - \mathrm{Re}(-z)^{1/2}\}$

for $z \in D(z_1, r, \varphi)$ [1, p. 101].

Let $z \in \overline{D}(z_1, r, \varphi)$. By means of the asymptotic formula (1) we get that

$$\frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} = \exp\{2n^{1/2}(\sqrt{-z} - \sqrt{-z_0})\} \frac{l^{(\alpha)}(z)}{l^{(\alpha)}(z_0)} \{1 + \frac{\lambda_n^{(\alpha)}(z) - \lambda_n^{(\alpha)}(z_0)}{1 + \lambda_n^{(\alpha)}(z_0)}\}.$$

From this equality and from (2) it follows that there exists a constant K_1 such that

$$\left| \frac{L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z_0)} \right| \leq K_1 \exp\{2n^{1/2}(\mathrm{Re}\sqrt{-z} - \lambda_0)\}$$

for $z \in \overline{D}(z_1, r, \varphi)$ and $n = 1, 2, \ldots$. The inequality (6) and this inequality yield

$$h_{s,2}(z) \le \varepsilon K_1 |z - z_1|^m \sum_{n=s+1}^{\infty} n^p \exp\{2n^{1/2}(\operatorname{Re}\sqrt{-z} - \lambda_0)\}$$

for $z \in D(z_1, r, \varphi)$.

Obviously

$$\sum_{n=s+1}^{\infty} n^p \exp\{2n^{1/2}(\operatorname{Re}\sqrt{-z} - \lambda_0)\} = O(\int_1^{+\infty} t^p \exp\{2t^{1/2}(\operatorname{Re}\sqrt{-z} - \lambda_0)\}dt).$$

It is easy to prove that

$$\int_1^{+\infty} t^p \exp\{-2t^{1/2}(\lambda_0 - \operatorname{Re}\sqrt{-z})\}dt \le K_2(\lambda_0 - \operatorname{Re}\sqrt{-z})^{-2p-2}, \quad z \in D(z_1, r, \varphi),$$

where K_2 is a constant. Then using (7) we obtain that

$$|z - z_1|^m \sum_{n=s+1}^{\infty} n^p \exp\{2n^{1/2}(\operatorname{Re}\sqrt{-z} - \lambda_0)\} \le K_3 |z - z_1|^{m-2p-2},$$

where K_3 is a constant. Since the inequality (5) holds and $z \in D(z_1, r, \varphi)$ there exists a constant K_4 for which

$$|z - z_1|^m \sum_{n=s+1}^{\infty} n^p \exp\{2n^{1/2}(\operatorname{Re}\sqrt{-z} - \lambda_0)\} \le K_4.$$

Therefore

(8) $$h_{s,2}(z) \le \varepsilon K_1 K_4.$$

for $z \in D(z_1, r, \varphi)$ and for $n > s_0$. Let such s be fixed, then there exists a positive constant K_5 such that

$$h_{s,1}(z) \le K_5 |z - z_1|^m$$

for $z \in D(z_1, r, \varphi)$. Moreover let $|z - z_1|^m < \varepsilon$. Then

(9) $$h_{s,1}(z) \le \varepsilon K_5.$$

From the inequalities (8) and (9) it follows that

$$h(z) \le \varepsilon(K_5 + K_1 K_4)$$

for $z \in D(z_1, r, \varphi)$ and sufficiently closed to z_1. This means that

$$\lim_{z \to z_1}(z - z_1)^m f(z) = 0, \qquad z \in D(z_1, r, \varphi).$$

However, this contradicts the assumption that the function $f(z)$ has a pole of order m at the point z_1. Thus Theorem 4 is proved.

Corollary. Let $\alpha > -1$, $z_0 \in G$ and $\lim_{n \to \infty} a_n L_n^{(\alpha)}(z_0) = 0$. If $f(z)$ is the sum of the series (3) in the region $\Delta(\lambda_0)$, where $\lambda_0 = \operatorname{Re}(-z_0)$, and $f(z)$ has a pole on the parabola $p(\lambda_0)$, then it is a simple pole.

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