LIE GROUPS AS 3-DIMENSIONAL ALMOST CONTACT B-METRIC MANIFOLDS IN TWO MAIN CLASSES

Miroslava Ivanova
Department of Informatics and Mathematics, Trakia University, Stara Zagora, 6000, Bulgaria,
e-mail: mivanova@uni-sz.bg

ABSTRACT

Three-dimensional almost contact B-metric manifolds are constructed by a three-parametric family of Lie groups. It is established the class of the investigated manifolds which has an important geometrical interpretation. It is determined also the type of the constructed Lie algebras in the Bianchi classification. There are given some geometric characteristics and properties of the considered manifolds.

Key words. Almost contact B-metric manifold; φ-holomorphic section; ξ-section; Lie group; Lie algebra.

Introduction

The geometry of the almost contact B-metric manifolds is the odd-dimensional counterpart of the geometry of the almost complex manifolds with Norden metric [3, 6]. The almost contact B-metric manifolds are introduced in [5]. These manifolds are investigated and studied for instance in [5, 12, 14, 15, 16, 18, 19, 21].

In this paper we focus our attention on the Lie groups considered as three-dimensional almost contact B-metric manifolds, bearing in mind the investigations in [10]. We construct a family of Lie groups and equip them with an almost contact B-metric structure from the direct sum of two of main classes. These classes are $\mathcal{F}_1$ and $\mathcal{F}_{11}$, where the fundamental tensor F is expressed explicitly by the structure (φ,ξ,η,g). The former class is the odd-dimensional analogue of the class of conformal Kähler manifolds with Norden metric and the latter class is the only basic class where the Lee form ω is not vanish whereas θ and θ^* are zero. Moreover, the metric g is represented on the $\mathcal{F}_1$-manifolds by its horizontal component $g(\varphi^2\cdot,\varphi^2\cdot)$, i.e. the restriction of g on the contact distribution $\ker(\eta)$, whereas on the $\mathcal{F}_{11}$-manifolds g is represented by its vertical component $\eta\otimes\eta$.

Our aim is to study some important geometric characteristics and properties of the obtained manifolds.

The paper is organized as follows. In Sec. 1, we give necessary facts about the considered manifolds. In Sec. 2, we construct a family of 3-dimensional Lie groups with almost contact B-metric structure and characterize it.

1 Almost contact manifolds with B-metric

Let (M,φ,ξ,η,g) be a $(2n+1)$-dimensional *almost contact B-metric manifold*, i.e. (φ,ξ,η) is an almost contact structure determined by a tensor field φ of type (1,1), a vector field ξ and its dual 1-form η as follows:

$$\varphi\xi=0,\quad \varphi^2=-\mathrm{Id}+\eta\otimes\xi,\quad \eta\circ\varphi=0,\quad \eta(\xi)=1,$$

where Id is the identity; moreover, g is a pseudo-Riemannian metric satisfying

$$g(\varphi x,\varphi y)=-g(x,y)+\eta(x)\eta(y)$$

for arbitrary tangent vectors $x,y\in T_pM$ at an arbitrary point $p\in M$ [5].

Further, x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$ or T_pM at arbitrary $p\in M$.

We note that the restriction of a B-metric on the contact distribution $H=\ker(\eta)$ coincides with the corresponding Norden metric with respect to the almost complex structure; the restriction of φ on H acts as an anti-isometry on the metric on H, which is the restriction of g on H. Thus, it is obtained a correlation between a $(2n+1)$-dimensional almost contact B-metric manifold and a $2n$-dimensional almost complex manifold with Norden metric. The associated metric $\tilde{g}$ of g on M is defined by $\tilde{g}(x,y)=g(x,\varphi y)+\eta(x)\eta(y)$ and is also a B-metric. The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is an almost contact B-metric manifold, too. Both metrics g and $\tilde{g}$ are indefinite of signature $(n+1,n)$.

The structure group of (M,φ,ξ,η,g) is $\mathcal{G}\times\mathcal{I}$, where $\mathcal{G}$ is the group $\mathcal{GL}(n;\mathbb{C})\cap\mathcal{O}(n,n)$ and $\mathcal{I}$ is the identity on $\operatorname{span}(\xi)$.

Let ∇ be the Levi-Civita connection of g. The tensor F of type (0,3) on M is defined by $F(x,y,z)=g((\nabla_x\varphi)y,z)$. It satisfies the following identities:

$$F(x,y,z)=F(x,z,y)=F(x,\varphi y,\varphi z)+\eta(y)F(x,\xi,z)+\eta(z)F(x,y,\xi).$$

Almost contact B-metric manifolds were classified in [5] into eleven basic classes $\mathcal{F}_i$ $(i=1,2,\dots,11)$ with respect to F. These eleven basic classes intersect in the special class $\mathcal{F}_0$ determined by the condition $F(x,y,z)=0$. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with ∇-parallel structures, i.e. $\nabla\varphi=\nabla\xi=\nabla\eta=\nabla g=\nabla\tilde{g}=0$.

Let $\{e_i\}_{i=1}^{2n+1}=\{e_1,e_2,\dots,e_{2n+1}\}$ be a basis of the tangent space T_pM of M at an arbitrary point $p\in M$. In this basis, let g_{ij} and g^{ij} be the components of the matrix of g and its inverse, respectively. The Lee forms θ, θ^* and ω associated with F are defined by:

$$\theta(z)=g^{ij}F(e_i,e_j,z),\quad \theta^*(z)=g^{ij}F(e_i,\varphi e_j,z),\quad \omega(z)=F(\xi,\xi,z).$$

The square norms of $\nabla\varphi$, $\nabla\eta$ and $\nabla\xi$ are defined by:

$$\begin{gathered}\|\nabla\varphi\|^2=g^{ij}g^{ks}g\left(\left(\nabla_{e_i}\varphi\right)e_k,\left(\nabla_{e_j}\varphi\right)e_s\right),\\ \|\nabla\eta\|^2=g^{ij}g^{ks}\left(\nabla_{e_i}\eta\right)e_k\left(\nabla_{e_j}\eta\right)e_s,\quad \|\nabla\xi\|^2=g^{ij}g\left(\nabla_{e_i}\xi,\nabla_{e_j}\xi\right),\end{gathered}\tag{1.1}$$

respectively ([13, 14]). Let us remark, if (M,φ,ξ,η,g) is an $\mathcal{F}_0$-manifold then the equality $\|\nabla\varphi\|^2=0$ is valid, but the inverse implication is not always true. According to [14], an almost contact B-metric manifold satisfying the condition $\|\nabla\varphi\|^2=0$ is called an *isotropic-*$\mathcal{F}_0$*-manifold*.

The curvature tensor R of type (1,3) of ∇ is defined by $R(x,y)z=\nabla_x\nabla_y z-\nabla_y\nabla_x z-\nabla_{[x,y]}z$. The corresponding tensor of R of type (0,4) is denoted by the same letter and is defined by $R(x,y,z,w)=g(R(x,y)z,w)$.

The Ricci tensor ρ and the scalar curvature τ for R as well as their associated quantities are defined as usual by $\rho(x,y)=g^{ij}R(e_i,x,y,e_j)$, $\tau=g^{ij}\rho(e_i,e_j)$, $\rho^*(x,y)=g^{ij}R(e_i,x,y,\varphi e_j)$ and $\tau^*=g^{ij}\rho^*(e_i,e_j)$, respectively.

In [20], an almost contact B-metric manifold (M,φ,ξ,η,g) is called *-Einstein if the Ricci tensor has the form $\rho=\alpha\tilde{g}|_H$, where α is a real constant.

If α is a non-degenerate 2-plane (section) in T_pM, $p\in M$, then, according to [21], the special 2-planes with respect to (φ,ξ,η,g) are: a *totally real section* if α is orthogonal to its φ

-image $\varphi\alpha$ and ξ, a φ*-holomorphic section* if α coincides with $\varphi\alpha$ and a ξ*-section* if ξ lies on α.

The sectional curvature $k(\alpha;p)(R)$ for R of α having an arbitrary basis $\{x,y\}$ at p is given by

$$k(\alpha;p)(R)=\frac{R(x,y,y,x)}{g(x,x)g(y,y)-g(x,y)^2}. \qquad (1.2)$$

In [14], a linear connection D is called a *natural connection* on an arbitrary almost contact B-metric manifold if its structure tensors are parallel with respect to D, i.e. $D\varphi=D\xi=D\eta=Dg=D\tilde{g}=0$. According to [19], D is natural on (M,φ,ξ,η,g) if and only if $D\varphi=Dg=0$. Natural connections exist on any almost contact B-metric manifold. They are restricted to ∇ only on an $\mathcal{F}_0$-manifold.

In [16], it is introduced a natural connection $\dot{D}$ on (M,φ,ξ,η,g) in all basic classes by

$$\dot{D}_x y=\nabla_x y+\frac{1}{2}\{(\nabla_x\varphi)\varphi y+(\nabla_x\eta)y\cdot\xi\}-\eta(y)\nabla_x\xi. \qquad (1.3)$$

This connection is called a φ*B-connection* in [17] and it is studied for the main classes $\mathcal{F}_1,\mathcal{F}_4,\mathcal{F}_5,\mathcal{F}_{11}$ in [16, 11, 12]. The φ B-connection is the odd-dimensional counterpart of the B-connection on the corresponding almost complex manifold with Norden metric, studied for the class $\mathcal{W}_1$ in [4].

In [18], a natural connection $\ddot{D}$ is called a φ*-canonical connection* on (M,φ,ξ,η,g) if its torsion tensor $\ddot{T}$, defined by $\ddot{T}(x,y,z)=g(\ddot{D}_x y-\ddot{D}_y x-[x,y],z)$, satisfies the following identity:

$$\begin{aligned}&\ddot{T}(x,y,z)-\ddot{T}(x,z,y)-\ddot{T}(x,\varphi y,\varphi z)+\ddot{T}(x,\varphi z,\varphi y)=\\&=\eta(x)\{\ddot{T}(\xi,y,z)-\ddot{T}(\xi,z,y)-\ddot{T}(\xi,\varphi y,\varphi z)+\ddot{T}(\xi,\varphi z,\varphi y)\}\\&+\eta(y)\{\ddot{T}(x,\xi,z)-\ddot{T}(x,z,\xi)-\eta(x)\ddot{T}(z,\xi,\xi)\}-\eta(z)\{\ddot{T}(x,\xi,y)-\ddot{T}(x,y,\xi)-\eta(x)\ddot{T}(y,\xi,\xi)\}.\end{aligned}$$

Moreover, there is proven that the φB-connection and the φ-canonical connection coincide on an almost contact B-metric manifold if and only if the manifold belongs to the class $\mathcal{F}_1\oplus\mathcal{F}_2\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_6\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$.

Another natural connection on (M,φ,ξ,η,g), which is called φKT-connection, is introduced and studied in [14]. This connection is defined by the condition its torsion to be a 3-form. There is proven that the φKT-connection exists if and only if (M,φ,ξ,η,g) belongs to $\mathcal{F}_3\oplus\mathcal{F}_7$. According to [7], the class of 3-dimensional almost contact B-metric manifolds is $\mathcal{F}_1\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$.

2 A family of Lie groups as 3-dimensional $(\mathcal{F}_1\oplus\mathcal{F}_{11})$-manifolds

Let G be a three-dimensional real connected Lie group and $\mathfrak{g}$ be its Lie algebra. Let $\{e_0,e_1,e_2\}$ be a global basis of left-invariant vector fields on G. We define an almost contact structure (φ,ξ,η) on G by

$$\begin{aligned}&\varphi e_0=o,\quad \varphi e_1=e_2,\quad \varphi e_2=-e_1,\quad \xi=e_0;\\&\eta(e_0)=1,\quad \eta(e_1)=\eta(e_2)=0,\end{aligned} \qquad (2.1)$$

where o is the zero vector field. We define a B-metric g on G by

$$g(e_0,e_0)=g(e_1,e_1)=-g(e_2,e_2)=1,\quad g(e_0,e_1)=g(e_0,e_2)=g(e_1,e_2)=0. \qquad (2.2)$$

We consider the Lie algebra $\mathfrak{g}$ on G, determined by the following non-zero commutators:

$$[e_0,e_1]=-de_0, \quad [e_0,e_2]=ce_0, \quad [e_1,e_2]=ae_1+be_2, \tag{2.3}$$

where $a,b,c,d\in\mathbb{R}$. We verify immediately that the Jacobi identity for this Lie algebra is satisfied if and only if the condition $ad=bc$ holds. By virtue of the latter equality, G is a three-parametric family of Lie groups.

Theorem 2.1 *Let (G,φ,ξ,η,g) be a three-dimensional connected Lie group with almost contact B-metric structure determined by (2.1), (2.2) and (2.3) with the condition $ad=bc$. Then it belongs to the class $\mathcal{F}_1\oplus\mathcal{F}_{11}$.*

Proof. The well-known Koszul equality of the Levi-Civita connection ∇ of g

$$2g(\nabla_{e_i}e_j,e_k)=g([e_i,e_j],e_k)+g([e_k,e_i],e_j)+g([e_k,e_j],e_i) \tag{2.4}$$

implies the following form of the components $F_{ijk}=F(e_i,e_j,e_k)$ of the tensor F:

$$2F_{ijk}=g([e_i,\varphi e_j]-\varphi[e_i,e_j],e_k)+g(\varphi[e_k,e_i]-[\varphi e_k,e_i],e_j)+g([e_k,\varphi e_j]-[\varphi e_k,e_j],e_i). \tag{2.5}$$

Using (2.5) and (2.3), we get the following non-zero components F_{ijk}:

$$F_{001}=F_{010}=c, \quad F_{002}=F_{020}=d, \quad F_{111}=F_{122}=2a, \quad F_{211}=F_{222}=-2b. \tag{2.6}$$

By direct verification, we establish that the components in (2.6) satisfy the condition $F=F^1+F^{11}$ for the components F^s of F in the basic classes $\mathcal{F}_s$ $(s=1,11)$ having the following form (see [7])

$$\begin{aligned}
&F^1(x,y,z)=(x^1\theta_1-x^2\theta_2)(y^1z^1+y^2z^2),\\
&\quad \theta_1=F_{111}=F_{122}, \quad \theta_2=-F_{211}=-F_{222};\\
&F^{11}(x,y,z)=x^0\{(y^1z^0+y^0z^1)\omega_1+(y^2z^0+y^0z^2)\omega_2\},\\
&\quad \omega_1=F_{010}=F_{001}, \quad \omega_2=F_{020}=F_{002},
\end{aligned} \tag{2.7}$$

where $\theta_i=\theta(e_i)$ and $\omega_i=\omega(e_i)$ $(i=1,2)$ are determined by $\theta_1=2a$, $\theta_2=2b$, $\omega_1=c$, $\omega_2=d$. Therefore, the manifold (G,φ,ξ,η,g) belongs to the class $\mathcal{F}_1\oplus\mathcal{F}_{11}$ of the mentioned classification.

Obviously, (G,φ,ξ,η,g) belongs to $\mathcal{F}_1$, $\mathcal{F}_{11}$ and $\mathcal{F}_0$ if and only if for all $i\in\{1,2\}$ the parameters θ_i, ω_i and $\theta_i=\omega_i$ vanish, respectively.

Bearing in mind the above, the commutators in (2.3) take the form

$$\begin{aligned}
&[e_0,e_1]=-\omega_2e_0, \quad [e_0,e_2]=\omega_1e_0, \quad [e_1,e_2]=\frac{1}{2}(\theta_1e_1+\theta_2e_2),\\
&\theta_1\omega_2=\theta_2\omega_1
\end{aligned} \tag{2.8}$$

in terms of the basic components of the Lee forms θ and ω.

In [22], it is consider the matrix Lie group G_I of the following form

$$G_I=\begin{pmatrix} e^{-z} & 0 & x\\ 0 & e^{z} & y\\ 0 & 0 & 1 \end{pmatrix},$$

where $x,y,z\in\mathbb{R}$. This Lie group is the group of hyperbolic motions of the plane $\mathbb{R}^2$. The corresponding Lie algebra is determined by the following commutators:

$$\mathfrak{g}_I: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2, \quad [X_1, X_2] = 0.$$

The type of this Lie algebra according to the Bianchi classification given in [1, 2] of three-dimensional real Lie algebras is Bia(V).

Bearing in mind [9], the matrix representation of the Lie algebra $\mathfrak{g}_I$ in the class $\mathcal{F}_1 \oplus \mathcal{F}_{11}$ is

$$A = \begin{pmatrix} -\omega_2 r + \omega_1 s & 0 & 0 \\ \omega_2 t & \frac{1}{2}\theta_1 q & \frac{1}{2}\theta_2 q \\ -\omega_1 t & -\frac{1}{2}\theta_1 p & -\frac{1}{2}\theta_2 p \end{pmatrix},$$

where $p, q, r, s, t \in \mathbb{R}$. Then substituting $X_1 = e_2$, $X_2 = e_0$, $X_3 = -e_1$, $\theta_1 = 0$, $\theta_2 = 2$, $\omega_1 = 0$, $\omega_2 = -1$ or $X_1 = e_1$, $X_2 = -e_0$, $X_3 = e_2$, $\theta_1 = 2$, $\theta_2 = 0$, $\omega_1 = -1$, $\omega_2 = 0$, we have that G_I can be considered as an almost contact B-metric manifold of the class $\mathcal{F}_1 \oplus \mathcal{F}_{11}$ [8].

Using (2.4) and (2.8), we obtain the components of ∇:

$$\begin{gathered} \nabla_{e_0} e_0 = \omega_2 e_1 + \omega_1 e_2, \quad \nabla_{e_0} e_1 = -\omega_2 e_0, \quad \nabla_{e_0} e_2 = \omega_1 e_0, \\ \nabla_{e_1} e_1 = \frac{1}{2}\theta_1 e_2, \quad \nabla_{e_1} e_2 = \frac{1}{2}\theta_1 e_1, \nabla_{e_2} e_1 = -\frac{1}{2}\theta_2 e_2, \quad \nabla_{e_2} e_2 = -\frac{1}{2}\theta_2 e_1. \end{gathered} \tag{2.9}$$

We denote by $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ the components of the curvature tensor R, $\rho_{jk} = \rho(e_j, e_k)$ of the Ricci tensor ρ, $\rho^*_{jk} = \rho^*(e_j, e_k)$ of the associated Ricci tensor ρ^* and k_{ij} of the sectional curvature for ∇ of the basic 2-plane α_{ij} with a basis $\{e_i, e_j\}$, where $e_i, e_j \in \{e_0, e_1, e_2\}$. On the considered manifold $(G, \varphi, \xi, \eta, g)$ the basic 2-planes α_{ij} are: φ-holomorphic section — α_{12} and ξ-sections — α_{01}, α_{02}. Further, by (??), (2.2), (2.8) and (2.9), we compute

$$\begin{gathered} R_{0110} = k_{01} = \frac{1}{2}\theta_1\omega_1 - \omega_2^2, \quad R_{0220} = -k_{02} = -\omega_1^2 + \frac{1}{2}\theta_2\omega_2, \\ R_{0120} = \rho_{12} = \frac{1}{2}\rho^*_{00} = \frac{1}{2}\tau^* = \left(\omega_1 - \frac{1}{2}\theta_1\right)\omega_2, \quad R_{1221} = -k_{12} = -\rho^*_{12} = -\frac{1}{4}(\theta_1^2 - \theta_2^2), \\ \rho_{00} = \frac{1}{2}(\theta_1\omega_1 - \theta_2\omega_2) + (\omega_1^2 - \omega_2^2), \\ \rho_{11} = \frac{1}{4}(\theta_1^2 - \theta_2^2) + \frac{1}{2}\theta_1\omega_1 - \omega_2^2, \quad \rho_{22} = -\frac{1}{4}(\theta_1^2 - \theta_2^2) + \frac{1}{2}\theta_2\omega_2 - \omega_1^2, \\ \tau = \frac{1}{2}(\theta_1^2 - \theta_2^2) + (\theta_1\omega_1 - \theta_2\omega_2) + 2(\omega_1^2 - \omega_2^2). \end{gathered} \tag{2.10}$$

The rest of non-zero components of R, ρ and ρ^* are determined by (2.10) and the properties $R_{ijkl} = R_{klij}$, $R_{ijkl} = -R_{jikl} = -R_{ijlk}$, $\rho_{jk} = \rho_{kj}$ and $\rho^*_{jk} = \rho^*_{kj}$.

Taking into account (??), we have

$$\|\nabla\varphi\|^2 = 2(\theta_1^2 - \theta_2^2 + \omega_1^2 - \omega_2^2), \tag{2.11}$$

$$\|\nabla\eta\|^2 = \|\nabla\xi\|^2 = -(\omega_1^2 - \omega_2^2). \tag{2.12}$$

Denoting $\varepsilon = \pm 1$, the following proposition is valid.

Proposition 2.2 *The following characteristics are valid for* (G,φ,ξ,η,g)*:*

1. The φB*-connection* $\dot{D}$ *(respectively,* φ*-canonical connection* $\ddot{D}$*) is zero in the basis* $\{e_0,e_1,e_2\}$*;*

2. The manifold is flat if and only if it belongs to either a subclass of $\mathcal{F}_1\oplus\mathcal{F}_{11}$ *determined by conditions* $\theta_1=\varepsilon\theta_2=2\omega_1=2\varepsilon\omega_2$ *or a subclass of* $\mathcal{F}_1$ *determined by* $\theta_1=\varepsilon\theta_2$*;*

*3. The manifold is Ricci-flat (respectively, *-Ricci-flat) if and only if it is flat.*

4. The manifold is scalar flat if and only if the conditions $\theta_1\pm\theta_2=\omega_1\pm\omega_2=0$ *are satisfied;*

*5. The manifold is *-scalar flat if and only if one of the following four conditions are satisfied:* $\theta_1-2\omega_1=\theta_2-2\omega_2=0$*,* $\theta_1=\omega_1=0$*,* $\theta_2=\omega_2=0$*,* $\omega_1=\omega_2=0$*;*

6. The manifold is an isotropic-$\mathcal{F}_0$*-manifold if and only if the conditions* $\theta_1\pm\theta_2=\omega_1\pm\omega_2=0$ *are satisfied.*

Proof. Using (1.3), (2.1) and (2.9), we get immediately the assertion (1). The assertions (2), (4) and (5) are obtained using (2.10). On the three-dimensional almost contact B-metric manifold with the basis $\{e_0,e_1,e_2\}$, bearing in mind the definition equalities of the Ricci tensor ρ and the ρ^*, we have

$$\rho_{jk}=R_{0jk0}+R_{1jk1}-R_{2jk2}\qquad \rho^*_{jk}=R_{1kj2}+R_{2jk1}.$$

By virtue of the latter equalities, we get the assertion (3). Bearing in mind (??), we establish the truthfulness of (6).

According to (2.10), (??) and (??), we obtain

Proposition 2.3 *The following properties are equivalent for the manifold* (G,φ,ξ,η,g)*:*

1. *it is an isotropic-*$\mathcal{F}_0$*-manifold;*
2. *it is scalar flat;*
3. *it is *-Einsteinian;*
4. *the dual vectors* Θ*,* Ω *of* θ*,* ω*, respectively, and the vector* $\nabla_\xi\xi$ *are isotropic;*
5. *the sectional curvatures* k_{ij} *vanish;*
6. *the equalities* $\theta_1\pm\theta_2=\omega_1\pm\omega_2=0$ *are valid.*

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