TWO-LEVEL REDUCING DEGENERACY MEASURE METHOD FOR 3-D SIMILICIAL MESHES

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ABSTRACT

Reducing the degeneracy measure is a crucial point in compiling sequences of successive triangulations. The reducing degeneracy measure procedures are very important in the isoparametric approach especially in the case of boundaries with turning lines. The usual refinement strategies produce sequences of successive triangulation with overspill property in this case. That is why the development of the improving quality procedures is well motivated. There are a lot of single grid techniques for improving the quality of meshes. The paper deals with an effective two-level multigrid method for reducing the measure of degeneracy. The method is applicable not only in affine case but also in isoparametric one for non Lipschitz domains. The numerical results indicate that the method is superior than the single grid ones.

Key words: *Sequences of finite element triangulations, optimal refinement strategies, isoparametric approach, curved boundaries*

Introduction

Every software which deals with triangulation of curved domains needs a procedure for controlling the measure of degeneracy. The roll of such procedures in isoparametric case is to avoid the overspill property [1] which makes the triangulations totally inapplicable. The most widespread application of the sequences of successive triangulations is in the multigrid solution methods for boundary and eigenvalue problems [9]. The reduction of the measure of degeneracy assure not only the convergence of the finite element solutions in a fixed triangulation but also the stability of intergrid operators [3]. There are various approaches for improving the quality of meshes. Such approaches are an object of a great interest in the last two decades, see for instance the works of J. Jouhaud, M. Montagnac and L. Tourrette [2], B. Sawicki and M. Okoniewski [7], D. Q. Ren [5] and D. Q. Ren, E. E. Bracken, S. V. Polstyanko, N. Lambert, R. Suda and D. D. Giannacopulos [6].

Too many authors consider polyhedral domains and affine finite elements or curved domains with subparametric elements. There are relatively few papers which deal with isoparametric elements. Most of the refinement strategies produce inapplicable sequence of successive triangulations in the case of non Lipschitz boundary.

An effective two-level multigrid method is obtained in the present paper. The method is applicable not only for triangulation of curved Lipschitz domains but also in the case when the boundary have turning lines. Computational tests indicates that the method reduces the measure of degeneracy thrice for domains with non Lipschitz boundaries. The algorithm is presented by pseudo code.

Sequences of successive triangulations

Let Ω be a three dimensional simply connected bounded domain with a boundary Γ. Consider a useful initial triangulation τ_0 of the domain Ω, which is as coarse as possible. Obtain a sequence of triangulations $\tau_1, \tau_2, \tau_{3,\dots,}$ by successive refinements of τ_0. The triangulations τ_k, $k=0,1,2,3,\dots$ are formed so that any two elements share at most a vertex, an edge, or a face. Moreover the adjacent elements have only shared nodes on their common faces (edges). Unifying all the elements in triangulation τ_k, we obtain an approximate domain $\Omega_k = \bigcup_{K \in \tau_k} K$ with a boundary Γ_k.

Definition 1 S. Zhang [10] *Introduce the measure of degeneracy of the finite element K by*

$$\delta(K) = \frac{h(K)}{\rho(K)}$$

where h(K) is the length of the longest edge of K and ρ(K) is the diameter of the biggest ball contained in K.

Then the measure of degeneracy for a triangulation τ_k and corresponding sequence of successive triangulations are determined as follows

$$\delta(\tau_k) = \min_{K \in \tau_k} \delta(K) \text{ and } \delta(\{\tau_k\}) = \inf_{k \in \mathrm{N} \cup \{0\}} \delta(\tau_k).$$

Denote the power of a refinement strategy A by

$$A^k K = A(A(A(\ldots A(K)\ldots))) - k\text{-times.}$$

The boundary layer B_k of any triangulation τ_k consists of those elements which have more than one vertex on the boundary. Separate the set B_k into B_{ki} $i=2,3$ where

B_{ki} ={the set of all elements which have i vertices on the boundary Γ_k}.

The two-level multigrid reducing degeneracy method

Let τ_k $\tau_k = A^k \tau_0(\Omega)$. Suppose that we obtained all members τ_i $i=0,1,2,\ldots,k$ of the sequences τ_n and should compile the next triangulation τ_{k+1}. Obviously it is impossible to improve all elements in any high level triangulation. That is why we introduce a set of elements M_k which should be improved in the k-th level.

The set M_k should consist of those elements of τ_{k+1} which have the worse measure of degeneracy.

The paper deals with curved domains that is why we consider the isoparametric case in details. The best choice in this case is $M_k = B_{k3}$. Then it is very important that any two elements from B_{k3} share no more than a curved edge. The 7-12 refinement strategy (7-12RS) [8] is very suitable in this case. Introduce floating edge nodes for all the elements of M_k. Find the best location of the floating nodes minimizing the degeneracy measure for all successors of each element of M_k. For this purpose we need fast and stable optimization algorithm. The computational tests of various optimization algorithms for nonsmooth functionals indicates that the superlinear convergent algorithm (BB) by J. Barzilai and J. Borwein [4] is appropriate for the optimization procedure. Of course the same problem could be solved by other optimization methods. The stability of the method for nonsmooth functionals is a crucial point regarding applications in our investigation. Quadratic and cubic isoparametric elements are the most used finite elements in engineering practice. For simplicity of exposition we restrict ourselves to sequences of successive triangulations obtained by 10-nodes quadratic finite elements.

Let the nodes $a_{iK} \in \Gamma_k$ $i=1,2,3$ and a_{7K}, a_{9K} and a_{10K} are the straight edge nodes for an element $K \in M_k$. Number the nodes in the finite element K so that $a_{7K} \in a_{1K}a_{4K}$, $a_{9K} \in a_{2K}a_{4K}$ and $a_{10K} \in a_{3K}a_{4K}$. Define

$$\begin{cases} a_{7K} = (a_{1K} - a_{4K})t_1 + a_{4K} \\ a_{9K} = (a_{2K} - a_{4K})t_2 + a_{4K} \\ a_{10K} = (a_{3K} - a_{4K})t_3 + a_{4K} \end{cases} . \tag{1}$$

Thus we obtain a finite element $K(\underline{t})\in M_k$ which depends on the vector $\underline{t}(t_1, t_2, t_3)$. The idea of the method is to minimize the function

$$\varphi(\underline{t})=\delta(AK(\underline{t})),\ \underline{t}\in T,$$

where

$$T=\{\underline{t}\ |\ 0\le t_i\le 1,\ i=1,2,3\}$$

is the unit cube.

Denote the point of the global minimum of $\varphi(\underline{t})$ in T by $\underline{t}$*. Then we replace the finite element K with $K^*=K(\underline{t}^*)$ $\forall K\in M_k$. Further we present a pseudo code implementation of the method.

Pseudo Code Implementation of the Two-Level Multigrid Reducing Degeneracy Method

program *The two-level algorithm*

procedure MD[$\underline{t}$, K]

Description {*The MD computes the measure of degeneracy of a finite element* $K\in\tau_k$. *The number* n_K *indicates the number of successors of the finite element K.*}

begin
  set the floating nodes as in (1);
  obtain $K_i=AK(\underline{t}),\ i=1,2,\ldots,n_K$;
  set $\varphi(\underline{t})=\delta(AK(\underline{t}))$
end

procedure DMD[$\underline{t}$, K]

Description {The DMD computes the approximate Fréchet derivative of the characteristic function $\delta(\square K(\underline{t}))$. *The parameter* $\underline{h}$ *is a three-dimensional vector with* $||\underline{h}||$ *small enough.}*

begin

$$D\varphi(\underline{t})=\frac{\varphi(\underline{t}+\underline{h})+\varphi(\underline{t}-\underline{h})}{2\,\|\underline{h}\|}$$

end

procedure Min[f]

Description {*The Min procedure determine the point of the global minimum of the function* $f(\underline{t})$ *when* $\underline{t}\in T$. *The MD and DMD are applied in the minimization procedure.*}

begin
  find $\underline{t}^*$: $f(\underline{t}^*)=\min_{\underline{t}\in T} f(\underline{t})$
end

begin{*The main algorithm starts from this point.*}
set an initial triangulation $\tau_0(\Omega)$ of the domain of interest;
for $k=0$ to n do
  begin
    forall $K\in M_k$ do
      begin
        $\underline{t}^*=\mathrm{Min}[\delta(AK(\underline{t}))]$;
        set K=K($\underline{t}$*)
      end;
    set $\tau_{k+1}=\square\ \tau_k$
  end

end

Numerical Example

Consider a domain Ω bounded by the astroidal ellipsoid

$$\sigma: \ x_1^{2/3} + x_2^{2/3} + x_3^{2/3} = a^{2/3}.$$

The boundary $\partial\Omega$ contains turning lines. For the sake of symmetry we can consider the domain

$$\Omega_1 \ : \ \begin{cases} x_1^{2/3} + x_2^{2/3} + x_3^{2/3} \le a^{2/3} \\ 0 \le x_i, \quad i = 1,2,3 \end{cases}.$$

The most qualitative initial triangulation can be obtained by refining the octahedron inscribed in **Ω**. Let μ_k= □□$^k\tau_0(\Omega)$ *(□ is the 7-12RS)* and τ_k be the corresponding sequence obtained by the two-level reducing degeneracy measure method. In the highest level Table 1 indicates that the measure of degeneracy is reduced thrice by the improving quality procedure.

Table 1. A comparison between measure of degeneracy with and without the effect of the two-level method.

Level	Cardinality	$\delta(\mu_k)$	$\delta(\tau_k)$
0	8	3.34607	3.34607
1	56	4.98070	4.01200
2	432	7.36446	5.24053
3	3424	13.20747	6.67898
4	27328	21.48473	10.04800
5	218496	36.35648	12.66793

Conclusion

A two-level multigrid method for reducing the measure of degeneracy is obtained. A structural algorithm for implementation of the method is presented in pseudo code. The method is especially effective for avoiding the overspill property when non Lipschitz domains are triangulated. The method can be successfully applied not only for compiling multigrid solution methods but also in engineering design. The numerical tests in domains with turning lines indicate that the measure of degeneracy is essentially reduced.

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