COMMUTATIONAL PROPERTIES OF OPERATORS OF MIXED TYPE PRESERVING THE POWERS - I

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ABSTRACT

The following operator of mixed type depending on two parameters p and q

$$Hy(z) = H_{p,q}y(z) = \frac{d^p}{dz^p}\left(z^q \int_0^z y(\zeta)d\zeta\right), \quad p,q \in \mathbb{N}_+.$$

is considered in the space A_0 of the functions analytic around the origin $z = 0$ in the complex plane $\mathbb{C}$. We study here the operator $H = H_{p,q} : A_0 \to A_0$ in the case when it *preserves* the powers, i.e. if $q - p + 1 = 0$. The author has considered the cases of *increasing* and *decreasing* the powers in the papers [2] and [3]. In this **part I** we prove that the operators L of the commutant $\{L \in A_0 \to A_0 : LH = HL\}$ of H have the form

$$Ly(z) = \sum_{k=0}^{\infty} \frac{y^{(k)}(0)}{k!} d_k z^k,$$

where $\{d_k\}_{k=0}^{\infty}$ is an arbitrary sequence of complex numbers.

***Key words**: commutant of linear operator, minimal commutativity*

Introduction

Let A_0 be the space of the functions analytic around the origin $z = 0$ in the complex plane $\mathbb{C}$. We want to consider an operator of mixed type by first integration $\int_0^z y(\zeta)d\zeta$, then multiplication by a non-negative power z^q and finally p times differentiation, i.e. we consider the following operator of mixed type

$$Hy(z) = H_{p,q}y(z) = \frac{d^p}{dz^p}\left(z^q \int_0^z y(\zeta)d\zeta\right), \quad p,q \in \mathbb{N}_+. \tag{1}$$

It is suitable to represent the action of the operator $H : A_0 \to A_0$ on a single power z^k when $k \geq p - q - 1$:

$$Hz^k = \frac{1}{k+1}(k+q+1)((k+q+1)-1)\ldots((k+q+1)-p+1)z^{(k+q+1)-p}.$$

The two parameters p and q, and especially the number $q - p + 1$, determine if the operator will increase, preserve, or decrease the powers.

If $\beta = q - p + 1 > 0$, then the operator H increases the powers by β:

$$Hz^k = b_k z^{k+\beta}; \quad b_k = \frac{1}{k+1} \cdot \frac{(k+q+1)!}{(k+\beta)!} \neq 0, \quad \beta = q - p + 1 > 0. \tag{2}$$

The author has considered the commutational properties of (2) in [2].

If $\beta < 0$, we can denote $\alpha = -\beta = p - q + 1 > 0$ and then the operator H decreases the powers by α

$$Hz^k = b_k z^{k-\alpha}; b_k = \begin{cases} \frac{1}{k+1} \frac{(k+q+1)!}{(k-\alpha)!} \neq 0 & k \geq \alpha = p - q - 1 > 0 \\ 0 & 0 \leq k \leq \alpha - 1. \end{cases} \quad (3)$$

The commutational properties of (3) were considered by the author in [3].

In this paper the case $\beta = q - p + 1 = 0$ will be considered, i.e. $p = q + 1$. Then the operator $H = H_{p,q}$ defined by (1) preserves the powers:

$$Hz^k = H_{p,q} z^k = b_k z^k; \quad b_k = \frac{1}{k+1} \cdot \frac{(k+q+1)!}{k!} \neq 0. \quad (4)$$

In fact, if $y(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function from A_0 with coefficients $a_k = \frac{d^{(k)}(0)}{k!}$, then we can use the short representation

$$Hy(z) = \sum_{k=0}^{\infty} a_k b_k z^k \quad (5)$$

with b_k from (4).

There are different particular cases of the operator $H = H_{p,q}$ given by (1) or (4) considered in the mathematical literature. Here we will mention as an example the Libera operator [4]

$$\frac{2}{z} \int_0^z y(t) dt,$$

when the parameters are $p = 0, q = -1$.

The operator $H = H_{p,q}$ can be considered also as an Hadamard product

$$Hy(z) = y(z) \circ b(z) = \left[\left(\sum_{k=0}^{\infty} a_k z^k \right) \circ \left(\sum_{k=0}^{\infty} b_k z^k \right) \right] = \sum_{k=0}^{\infty} a_k b_k z^k,$$

and to consider such operator as a generalized integration, one has to assume $\lim_{k \to \infty} b_k = 0$ (see Samko S. G., A. A. Kilbas, O. I. Marichev [5, Sect. 22].

Let us present now the definition of commutant:

Definition 1:
It is said that a continuous linear operator L *commutes with a fixed operator* H, if $LH = HL$. The set of all such operators is called the *commutant* of H and will be denoted by C_H.

Description of the commutant

The following theorem describes the commutant C_H of the operator H.

Theorem 1:
Let $H : A_0 \to A_0$ be the general operator defined by (1) with $q - p + 1 = 0$, i.e. by (4). Then a linear operator $L : A_0 \to A_0$ belongs to the commutant C_H of H if and only if it has the form

$$Ly(z)=\sum_{k=0}^{\infty}\frac{y^{(k)}(0)}{k!}d_k z^k, \tag{6}$$

where $\{d_k\}_{k=0}^{\infty}$ is an arbitrary sequence of complex numbers, but such that the series in (6) is convergent.

Proof: Let $y(z)=\sum_{k=0}^{\infty}c_k z^k \in A_0$ with $c_k=\dfrac{y^{(k)}(0)}{k!}$, and let the power series expansion of $Ly(z)$ for an arbitrary operator L of the commutant C_H be

$$Ly(z)=\sum_{k=0}^{\infty}c_k\sum_{j=0}^{\infty}\lambda_{k,j}z^j \tag{7}$$

with unknown coefficients $\lambda_{k,j}$. In the particular case $y(z)=z^k$ we can use

$$Lz^k=\sum_{j=0}^{\infty}\lambda_{k,j}z^j. \tag{8}$$

Now instead of

$$LHy(z)=HLy(z) \tag{9}$$

we can consider the commutational property

$$LHz^k=HLz^k, \qquad k=0,1,2,\dots \tag{10}$$

for arbitrarily fixed k in order to find the numbers $\lambda_{k,j}$, since the powers z^k, $k=0,1,2,\dots$, form a basis of A_0.

Let us express LHz^k and HLz^k using (4) and (8):

$$LHz^k=L\left(b_k z^k\right)=b_k Lz^k=b_k\sum_{j=0}^{\infty}\lambda_{k,j}z^j=\sum_{j=0}^{\infty}\lambda_{k,j}b_k z^j \tag{11}$$

$$HLz^k=H\left(\sum_{j=0}^{\infty}\lambda_{k,j}z^j\right)=\sum_{j=0}^{\infty}\lambda_{k,j}Hz^j=\sum_{j=0}^{\infty}\lambda_{k,j}b_j z^j. \tag{12}$$

Equating the coefficients of the equal powers in (11) and (12), we have $\lambda_{k,j}b_k=\lambda_{k,j}b_j$, and using $b_k\neq b_j$ for $k\neq j$, we can express

$$\lambda_{k,j}=\begin{cases}0 & j\neq k\\ d_k & j=k,\end{cases} \tag{13}$$

where d_k, $k=0,1,2,\dots$, can be arbitrarily chosen complex numbers. Thus Lz^k can be written as

$$Lz^k=d_k z^k, \qquad d_k \text{ - arbitrary}, \tag{14}$$

and then

$$Ly(z)=\sum_{k=0}^{\infty}c_k d_k z^k, \tag{15}$$

shows that the desired description (6) is a *necessary* condition for L to commute with H.

The *sufficiency* of (6) or (15) follows immediately:

$$LHy(z)=L\left(\sum_{k=0}^{\infty}c_k Hz^k\right)=L\left(\sum_{k=0}^{\infty}c_k b_k z^k\right)=\sum_{k=0}^{\infty}c_k b_k Lz^k=$$

$$= \sum_{k=0}^{\infty} c_k b_k d_k z^k = \sum_{k=0}^{\infty} c_k d_k H z^k = H\left(\sum_{k=0}^{\infty} c_k d_k z^k\right) = HLy(z) \tag{16}$$

and the theorem is proved.

Remark:
Let us note that a sufficient condition for convergence of the series in the description (6) or (15) in the space A_0 is $\limsup_{k\to\infty} \sqrt{|d_k|} < \infty$. To prove this, we have to use the Cauchy-Hadamard formula for the radius of convergence and have to show that

$$R_L = \frac{1}{\limsup_{k\to\infty} \sqrt{|c_k d_k|}} > 0 \quad \text{or} \quad \limsup_{k\to\infty} \sqrt{|c_k d_k|} < \infty.$$

But $y(z) = \sum_{k=0}^{\infty} c_k z^k$ has a non-zero radius of convergence and

$$R_y = \frac{1}{\limsup_{k\to\infty} \sqrt{|c_k|}} > 0 \quad \text{or} \quad \limsup_{k\to\infty} \sqrt{|c_k|} < \infty.$$

Finally,

$$\limsup_{k\to\infty} \sqrt{|c_k d_k|} < \infty \le \limsup_{k\to\infty} \sqrt{|c_k|}.\limsup_{k\to\infty} \sqrt{|d_k|} < \infty$$

ensures the convergence in (6) or (15).

Let us announce here that in **part II** of this paper we will prove the *minimal commutativity* of the operator $H = H_{p,q}$. The following two definitions are given by Raichinov in [1]:

Definition 2:
It is said that a continuous linear operator T is *generated by an operator* H, if T is a finite or infinite sum

$$T = \sum_{n=0}^{\infty} d_n H^n,\ d_n \in \mathbb{C}.$$

Every operator T generated by H commutes with H, but the opposite is not true in general. Therefore the following definition is natural:

Definition 3:
An operator H is called *minimally commutative* if its commutant consists only of operators T generated by H.

The following theorem will be proved in **part II** of this paper:

Theorem 2.
If the operator H defined by (1) or (4) is considered in the subspace $S \subset A_0$ of the polynomials, then it is finitely minimally commutative.

To make this statement completely clear, let us mention that the minimal commutativity can be treated in two different ways, namely *finite* and *infinite* minimal commutativity. If the commutant C_H contains only elements of the form

$$A_n = \sum_{j=0}^{n} a_j H^j, \ a_j \in \mathbb{C},$$

with finite sums, then H is called *finitely* minimally commutative.

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