CHARACTERIZATION OF THE RIEMANNIAN *k*-POINTWISE OSSERMAN MANIFOLDS

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ABSTRACT

In the present short note we characterize the four-dimensional Riemannian k-pointwise Osserman manifolds. This is the class of the Riemannian manifolds for which the Jacobi operator from order k, defined from Gilkey, Stanilov and Videv, has a pointwice constant eigenvalues at any point of the manifold .

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Let (M,g) be an m-dimensional Riemannian manifold with a metric tensor g and curvature tensor R, let p be a point of M, and let M_p be the tangent space to the manifold at this point. We have the following well-known definitions:

Definition 1. *Let (M,g) be an m-dimensional Riemannian manifold, let p be a point of M , and let E^k be an κ-dimensional tangent subspace*($k<m$) *of the tangent space M_p, to M, at this point. If $e_1,e_2,\dots,e_k$ is an orthonormal basis in E^k, then the curvature operator*

(1) $$R(E^k)(u)=\sum_{i=1}^{k}R(u,e_i,e_i),\ i=1,2,\dots k,.$$

was called generalized Jacobi operator from order k(*which doesn't depend on the orthonormal basis $e_1,e_2,\dots,e_k$)*[1], *and when k*=1 *was called Jacobi operator*[3].

Definition 2. *Let (M,g) be an m-dimensional Riemannian manifold*, *such that eigenvalues of any Jacobi operator $R(E^k)$ from order k*, *are a pointwise function at any point $p\in M$. Then (M,g) was called pointwice k-Osserman manifold* [1], *and when k*=1, *pointwise Osserman manifold* [3].

Definition 3. *Let (M,g) be an m-dimensional Riemannian manifold*, *such that eigenvalues of any Jacobi operator $R(E^k)$ from order k*, *are globally constants at any point $p\in M$. Then (M,g) was called globally k-Osserman manifold and when k*=1[1], *globally Osserman manifold* [3].

From the algebraic results in[1] we get

Theorem 1. *Let (M,g) be an m-dimensional pointwise k-Osserman Riemannian manifold(*$1<k<m-1$). *Then if m is odd number, then there exist a pointwise constant K_0 , such that the curvature tensor R of the manifold (M,g) has the form*

(2) $$R(x,y,z)=K0.(g(y,z)x-g(x,z)y) .$$

If m is an even number, then there exist an almost complex structure J on M, *such that* (M,g,J) *is an almost Hermitian manifod*[4] *with curvature tensor of the form*:

(3) $$R(x,y,z)=KJ.(g(Jx,y)Jz+g(Jz,x)Jy-2g(x,Jy)Jz).$$

Our main result is:

Theorem 2. *An m-dimensional Riemannian manifold* (M,g) *is a pointwise k-Osserman manifold* ($1< k<m-1$) *if and only if there exist an almost complex structure J and a pointwise constant K_J, such that (M,g,J) is a four-dimensional almost Hermitian manifold with curvature tensor of the form* (3).

Proof. Suppose *(M,g)* is a k-pointwise Osserman manifold, and let m *is* an odd number, then according to theorem 1, for the curvature tensor R of the manifold *(M,g)* we have expression (2). From here it follows that any Jacobi operator $R(e_i)$ has the form

$$R(e_i)(u)=K_0(u-g(u,e_i)\,e_i)$$

for any of the tangent vectors $e_1, e_2,\dots, e_n$, of any orthonormal basis in the tangent space M_p , to the manifold M, at a point $p\in M$. Also any Jacobi operator $R(E^k)$ from order $k<m-1$, has the form:

$$R(E^k)=\sum_{i=1}^{k}K_0(u-g(u,e_i)e_i)$$

and from here we get that the orthonormal basis $e_1,e_2,\ldots,e_k$ in the tangent subspace E^k , is an eigenvector basis for the curvature operator $R(E^k)$, with the corresponding eigenvalues 0, $kK_{0.}$, kK_0, kK_0 , which are a poitwise functions on the manifold *(M,g)*. Since $m\geq 3(k\geq 2)$, then according to the Shour's theorem[6] we have that all eigenvalues, for any Jacobi operator $R(E^k)$ of order k, are a globally constants on the manifold (M,g), and (M,g) is a globally k-Osserman manifold, which is not pointwise k-Osserman.

Suppose *(M,g)* is a k-pointwise Osserman and let $m(1<k<m-1)$ is an even number. Then according to the theorem 1, there exist an almost complex structure J on M, such that (M,g,J) is an almost Hermitian manifold with curvature tensor R of the form (3). Let be $e_1,e_2,\ldots,e_{m/2}$, $Je_{m/2+1}$, $Je_{m/2+2},\ldots, Je_m$ be an arbitrary adapted orthonormal basis in the tangent space M_p, to the manifold M, at a point $p\in M$. Let E^k be an arbitrary non-degenerated subspace in M_p (the case when E^k is degenerated is trivial[1]), and let $e_1,e_2,\ldots,e_k$ be an orthonormal basis in E^k, which is chosen from $e_1,e_2,\ldots,e_{m/2},Je_{m/2+1},Je_{m/2+2},\ldots,Je_m$. Then we can construct Jacobi operator $R(E^k)$ from order k, and in fact this way we can consider any Jacobi operator $R(E^k)$ from order k, for any non-degenerated tangent subspace E^k in the the tangent space M_p, at a point $p\in M$. Now from the curvature representations (3), of the curvature tensor R, of the manifold (M,g,J), we have the following Jacobi operators:

$$R(e_i)(x)=K_J.((g(Jx,e_i)-g(x,Je_i))\,Je_i\,,$$
$$R(Je_i)(x)=-K_J.((g(Jx,Je_i)+g(x,e_i))e_i\,,$$

Then for any Jacobi operator $R(E^k)$ from order k, for any tangent subspace E^k in the the tangent space M_p, at a point $p\in M$ we have expressions:

(4)

$$R(E^k)(x)=K_J\Big(\sum_{i=1}^{\frac{k}{2}}(g(Jx,e_i)-g(x,Je_i))Je_i-\sum_{i=1}^{\frac{k}{2}}(g(Jx,Je_i)-g(x,e_i))e_i\Big),$$

if k is even, and

(5)

$$R(E^k)(x)=K_J\Big(\sum_{i=1}^{\frac{k}{2}+1}(g(Jx,e_i)-g(x,Je_i))Je_i-\sum_{i=1}^{\frac{k}{2}}(g(Jx,Je_i)-g(x,e_i))e_i\Big),$$

when k is an odd number. From here we get that in the case (4), the tangent vectors $e_1,e_2,\ldots,e_k$ $,Je_1,Je_2,\ldots,Je_k$ are an eigenvectors of the Jacobi operator $R(E^k)$ from order k, with the corresponding eigenvalues $0,0,\ldots,0,-2K_{J.}$, $-2K_J$,…, $-2K_J$, for any tangent subspace E^k in the tangent space M_p, at any point $p\in M$. Also in the case (5) the tangent vectors $e_1,e_2,\ldots,e_k,e_{k+1}$ $,Je_1,Je_2,\ldots,Je_k$ are an eigenvectors of the Jacobi operator $R(E^k)$ from order k, with the corresponding eigenvalues 0,0, $\ldots,0,0,-2K_{J.},-2K_J,\ldots,-2K_J$, for any tangent subspace E^k in the tangent space M_p, at any point $p\in M$. From these two cases, when m is even, we obtain that if (M,g) is a pointwise k-Osserman manifold, then all eigenvalues of any Jacobi operator $R(E^k)$ from order k , for any tangent subspace E^k in the tangent space M_p, at any point $p\in M$, are equal to 0 or -2 K_J , where K_J is a pointwise function of the manifold M. But in the case $m>4$, according to the well known result in [5], we have that K_J is a globally constant on the manifold (M,g,J) and then (M,g,J) is a Käeler manifold of constant holomorphic sectional curvature. Having in mind curvature representation in this case and (3), we obtain that $K_J=0$, which means that *(M,g,J)* is flat manifold and hence it is a trivial case for the globally k-Osserman manifold, which is not pointwise Osserman. Thus from our assumption *(M,g)* to be an m-dimensional pointwise k-Osserman Riemannian manifold, we get that it is possible only in the case when dim $M=m=4$ and in this case, there exist an almost complex structure J on M, such that (M,g,J) is an almost Hermitian manifold with curvature tensor R of the form (3). Conversely if (M,g,J) is an almost Hermitian manifold with curvature tensor R of the form (3), then all non-zero eigenvalues of any Jacobi operator $R(E^k)$ from order k, for any non-degenerated tangent subspace E^k , in the tangent space M_p, are pointwise function equal to $-2K_J$, at any point $p\in M$, and hence (M,g,J) is a k-pointwise Osserman manifold.

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