ON SOME JACOBI SERIES HAVING POLAR SINGULARITIES

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ABSTRACT

The present paper contains properties of some Jacobi series having polar singularities.
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Let α and β be a complex numbers such that α, β and $\alpha+\beta+1$ are not equal to $-1,\ -2,\ \ldots$. The polynomials $\{P_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ defined by equalities

$$P_n^{(\alpha,\beta)}(z)=\binom{n+\alpha}{n}{}_2F_1(-n,\ n+\alpha+\beta+1,\ \alpha+1;\ \frac{1-z}{2}),\ n=0,\ 1,\ 2,\ \ldots;\ z\in\mathbf{C},$$

where $\mathbf{C}$ is the complex plane and ${}_2F_1(a,b,c;\ \zeta)$ is Gauss hypergeometric function, are called Jacobi polynomials with parameters α and $\boldsymbol{\beta}$. The functions $\{Q_n^{(\alpha,\beta)}(z)\}_{n=0}^{+\infty}$ defined by equalities [1, Chapter I, (5.17)]

(1) $Q_n^{(\alpha,\beta)}(z)=$

$$\frac{2(n+\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(z-1)^{n+1}}\,{}_2F_1(n+1,\ n+\alpha+1,\ 2n+\alpha+\beta+1;\ \frac{2}{1-z}),$$

$n=0,\ 1,\ 2,\ \ldots;\ z\in G=\mathbf{C}\setminus[-1,\ 1]$,

are called Jacobi associated functions.

If α, β and n are fixed then from (1) it follows that

(2) $$|Q_n^{(\alpha,\beta)}(z)|=O(|z|^{-n-1})\qquad |z|\to+\infty.$$

Let $\omega(z)$ is that inverse of Zhukovsky function in the region G for which $|\omega(z)|>1$. Then in the region G the Jacobi polynomials and Jacobi associated functions have respectively the representations $(n\geq 1)$ [1, Chapter III, (1.9), (1.30)]

$$P_n^{(\alpha,\beta)}(z)=P^{(\alpha,\beta)}(z)n^{-\frac{1}{2}}[\omega(z)]^n\{1+p_n^{(\alpha,\beta)}(z)\},$$

and

(3) $$Q_n^{(\alpha,\beta)}(z)=Q^{(\alpha,\beta)}(z)n^{-\frac{1}{2}}[\omega(z)]^{-n-1}\{1+q_n^{(\alpha,\beta)}(z)\}$$

where $P^{(\alpha,\beta)}(z)\neq 0$, $Q^{(\alpha,\beta)}(z)\neq 0$, $\{p_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$ and $\{q_n^{(\alpha,\beta)}(z)\}_{n=1}^{+\infty}$ are holomorhpic functions in the region G.

If $n\to+\infty$ then $p_n^{(\alpha,\beta)}(z)=O(n^{-1})$ and $q_n^{(\alpha,\beta)}(z)=O(n^{-1})$ uniformly on every compact subset of G.

The series of kind

(4) $$\sum_{n=0}^{+\infty}a_nP_n^{(\alpha,\beta)}(z)$$

is called Jacobi series.

If

$$0<r^{-1}=\limsup_{n\to+\infty}|a_n|^{\frac{1}{n}}<1,$$

then the series (4) is absolutely uniformly convergent on every compact subset of region $E(r): \{z \in \mathbf{C} : |z+1| + |z-1| < r + r^{-1}\}$ and divergent in $\mathbf{C} \setminus \overline{E(r)}$ [1, (IV.1.1), (b)].

Theorem 1.[1, (V.1.2)] Let α and β be a complex numbers such that α, β and $\alpha+\beta+1$ are not equal to $-1, -2, \ldots$ and $R > 1$. If $f(z)$ is a complex function holomorphic in the region $E(R)$, then $f(z)$ is representable in $E(R)$ by a series of kind (1), i.e.

$$f(z) = \sum_{n=0}^{+\infty} a_n P_n^{(\alpha,\beta)}(z), \ z \in E(R)$$

with coefficients

(5) $$a_n = \frac{1}{2\pi i I_n^{(\alpha,\beta)}} \int_{\gamma(r)} f(\varsigma) Q_n^{(\alpha,\beta)}(\varsigma) d\varsigma, \ 1 < r < R, \ n = 0, 1, 2, \ldots,$$

where

(6) $$I_n^{(\alpha,\beta)} = \begin{cases} \dfrac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \ n = 0 \\ \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \ n \ge 1 \end{cases}$$

Theorem 2. Suppose that $\alpha, \beta, \alpha+\beta+1 \ne -1, -2, \ldots$, $1 < R < +\infty$ and $f(z)$ is a meromorphic function in **C**. Moreover let $f(z)$ have only simple poles at the points $z_1, z_2, \ldots, z_m \notin E(R)$ and

(7) $$\int_{\gamma(\sigma)} |f(z)| ds = O(1) \ (\sigma > 1, \ \sigma \to +\infty).$$

Then

(8) $$a_n = -\frac{1}{I_n^{(\alpha,\beta)}} \sum_{k=1}^{m} A_k Q_n^{(\alpha,\beta)}(z_k),$$

where $A_k = \operatorname{Res}_{\zeta=z_k} f(\zeta) \, (k = 1, 2, \ldots, m)$ and n is fixed.

Proof. Let $r_k = |w(z_k)| \, (k = 1, 2, \ldots, m)$ and $r_1 \le r_2 \le \ldots \le r_m$. Obviously $r_1 = R$. Let $2\rho > \max_{1\le k\le m}(r_k + r_k^{-1})$ and $1 < r < R$. Using residue theorem we get

$$\int_{|\zeta|=\rho} f(\zeta) Q_n^{(\alpha,\beta)}(\zeta) d\zeta - \int_{\gamma(r)} f(\zeta) Q_n^{(\alpha,\beta)}(\zeta) d\zeta = 2\pi i \sum_{k=1}^{m} \operatorname{Res}_{\zeta=z_k} \{f(\zeta) Q_n^{(\alpha,\beta)}(\zeta)\}.$$

From (2) and (7) it follows that

$$\lim_{\rho\to+\infty} \int_{|\zeta|=\rho} f(\zeta) Q_n^{(\alpha,\beta)}(\zeta) d\zeta = 0.$$

Then

$$\int_{\gamma(r)} f(\zeta) Q_n^{(\alpha,\beta)}(\zeta) d\zeta = -2\pi i \sum_{k=1}^{m} \operatorname{Res}_{\zeta=z_k} \{f(\zeta) Q_n^{(\alpha,\beta)}(\zeta)\}.$$

Since $z_1, z_2, \ldots, z_m$ are simple poles of $f(z)$ and $Q_n^{(\alpha,\beta)}(z)$ is a holomorphic function in G we get that

$$\operatorname{Res}_{\zeta=z_k} \{f(\zeta) Q_n^{(\alpha,\beta)}(\zeta)\} = A_k Q_n^{(\alpha,\beta)}(z_k), \ k = 1, 2, \ldots, m.$$

So we get that

$$\int_{\gamma(r)} f(\zeta)Q_n^{(\alpha,\beta)}(\zeta)d\zeta = -2\pi i\sum_{k=1}^{m} A_k Q_n^{(\alpha,\beta)}(z_k).$$

Using this equality and (5) we come to the equality (8) and thus Theorem 2 is proved.

Now we consider a corollary of Theorem 2.

Let us recall that [1, Appendix, (1.14)]

$$\frac{\Gamma(n+u)}{\Gamma(n+v)} = n^{u-v}\{1+O(\frac{1}{n})\}, \quad n \to +\infty,$$

where u and v are arbitrary complex number. Using this asymptotic formula it is easy to prove that

(9) $$I_n^{(\alpha,\beta)} = O(\frac{1}{n}), \quad n \to +\infty.$$

Using (3) we obtain that

$$|\sum_{k=1}^{m} A_k Q_n^{(\alpha,\beta)}(z_k)| = O\{n^{-1/2}\sum_{k=1}^{m} r_k^{-n}\}, \quad n \to +\infty.$$

Since $1 < R = r_1 \le r_2 \le \ldots \le r_m$ then

$$\sum_{k=1}^{m} r_k^{-n} \le R^{-n}\sum_{k=1}^{m}\left(\frac{R}{r_k}\right)^n \le mR^{-n}.$$

Obviously

$$|\sum_{k=1}^{m} A_k Q_n^{(\alpha,\beta)}(z_k)| = O\{n^{-1/2}R^{-n}\}, \quad n \to +\infty.$$

Using this asymptotic formula, asymptotic formula (9) and representation (8) we get that

(10) $$|a_n R^n| = O(n^{1/2}), \quad n \to +\infty.$$

Finally we formulate a statement for Jacobi series whose coefficients satisfy this asymptotic formula.

Theorem 3. Suppose that $\alpha, \beta, \alpha+\beta+1 \ne -1, -2, \ldots$ and $1 < R < +\infty$. Let the coefficients $\{a_n\}_{n=0}^{+\infty}$ of Jacobi series (4) satisfy the condition (10) and $f(z)$ is the sum of (4) in $E(R)$. Then every pole of $f(z)$ on the ellipse $\gamma(R)$ is simple.

References

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